The Logic of Imaginary Scenarios

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Abstract

Imagining is something we use everyday in our lives, and in a wide variety of ways. In spite of the amount of works devoted to its study from both psychology and philosophy, there are only a few formal systems capable of modeling it; besides, almost all of those systems are static, in the sense that their models are initially predefined, and they fail to capture the dynamic process behind the creation of new imaginary scenarios. In this work, we review some influential theories of imagination and use their insights to distill an algorithm describing such process. Then, we use this algorithm to define a dynamic logical system built upon on a single-agent epistemic logic that provides the necessary tools to capture how the agent voluntarily creates new imaginary worlds; in other words, our system allows the model to be expanded dynamically at any time as a result of the agent performing an act of imagination. Furthermore, we provide an axiomatization and prove that the system is sound and complete.

Keywords: *imagination logic, hybrid logic, dynamic logic, dynamic imagination, imaginary worlds.*

1 Introduction and Motivation

Imagining is something we use everyday in our lives, and in a wide variety of ways: when planning our next move in a chess game, when picturing how we could decorate our new room, or even when listening to a story-teller, our mind creates, develops and evaluates imaginary worlds aimed to guide our actions, update our beliefs, or simply entertain us. Imagination has received a great deal of attention by philosophers, cognitive scientists and psychologists (as it can be seen in works like [23], [11], [19], or [13], among others). Its interest within the studies of the mind is beyond any doubt, and its relation to other mechanisms of the mind, such as emotions, behavior, desires and beliefs, makes imagination particularly interesting in many different areas.

Many authors distinguish between different mental attitudes related to imagination. For instance, [12] or [20] distinguish "imagination" and "pretense" by requiring to the latter attitude behavior and action, whereas other works like [2] draw a distinction between "imagine", "suppose" and "conceive", which are seen as three different ways to refer to our ability to think about scenarios and objects that may or may not exist.

The main goal of the present paper is to define a dynamic formal system that allows to represent, through the execution of an algorithm and the expansion of their formal models, how an agent creates new *imaginary scenarios* (i.e.; a new, or a set of new imaginary worlds) as a result of executing a voluntary act of imagination. There are two important clarifications that should be kept in mind throughout the rest of the present paper, and which help understand both the way we use certain terms, and the overall aim of our work:

1. When we talk about an "imaginary scenario", or an "imaginary world" we refer to any mental representation of a state of affairs[1] that is not actual and, moreover, about which the relevant agent is aware of it not being actual. Therefore, representations of state of affairs used in supposition, or pretense, for instance, are also taken into account in our work, and we refer to them as imaginary worlds as well.

[1] As we will introduce later, we use a possible-worlds semantics to formally represent such states of affairs.

2. Our main goal lies in providing a formal logical system that models how imaginary scenarios are created, but we do not consider, in our study, what happens when such worlds have been created, or why the agent is creating them, or what outcomes follow from the agent entertaining them[2]. Therefore, we do not restrict ourselves to any particular kind of imagination act, like supposition, conception or pretense. As each and every one of these mental actions involve creating and elaborating representations of state of affairs that are not actual, we consider them all indistinctly. Therefore, we will be talking about "imagination" in a broad sense that includes all those more fine-grained kinds of imagination acts.

This paper is outlined as follows. We review some influential theories of imagination, existing logical systems and identify the structure of an algorithmic account of imagination acts in Section 2. In Section 3, we define the syntax, formal models, algorithm and semantics for the Logic of Imaginary Scenarios; furthermore, we provide an example that shows how out system works. Once the formal details are set, we provide an axiomatization for the logic and prove that it is sound and complete in Section 4. Last but not least, we provide a discussion in Section 5, and we we conclude and point towards some lines of future work in Section 6.

2 Distilling the Dynamics of Imagination

Our approach to a formal treatment of the dynamics of imagination is based, mainly, on the account given by Peter Langland-Hassan in [14] about the processes involved in what he calls *Guiding Chosen* (GC) imaginings. The quasi-formal treatment of Langland-Hassan's work towards the processes of imagination makes it a good starting point for our interest in drawing bridges between the dynamics of imagination and formal systems. In addition, we also draw a structural comparison with other procedural theories of imaginary scenarios, such as Shaun Nichols and Stepehen Stich's *cognitive theory of pretense* [20], or Timothy Williamson's work on voluntary and involuntary modes of imagination [25].

Our main focus, with regards to imagination, is on *voluntary* imagination acts; in a nutshell, this kind of imagination acts are those that are willingly and consciously initiated by the agent via deciding to imagine such-and-such. Conversely, involuntary acts of imagination would be those in which an imagining spontaneously pops up into the mind of the agent without her intention of initiating it. Both our interest and the theories we review in the following paragraphs focus on this kind of imaginings.

Langland-Hassan's account involves the use of three distinct processes: the initial involvement of *top-down intentions* that initiate an imagining, the use of *lateral constraints* in the development of such imagining, and the *cyclical involvement of top-down intentions* that are used to add new premises during the imagining. The so-called top-down intentions are voluntary, conscious actions of the agent regarding *what* to imagine that define the content of the newly-created imagining. As soon as the initial content is set up, a set of lateral constrains encoding the algorithms, norms and rules specifying how the imagined scenario would likely unfold are used to elaborate on the details of what else would be the case, given the initial conditions. The author refers to the *deviance objection* to name those cases in which something that would not naturally follow from the scenario is still added into the imagining; in those cases, Langland-Hassan identifies in this a voluntary addition of new content that is clamped into the imagining. These additions are seen by the author as new top-down intentions that begin the cycle anew, and thus repeat the process of clamping some new conditions to the imagining, and then letting the lateral constrains unfold whatever would follow from them.

The previous theory follows up from the work from Nichols and Stich in [20], and aims to cover some of its caveats. Structurally speaking, Nichols and Stich's theory is quite similar. The authors identify a first, voluntary addition of an initial premise into what they call the *Possible-World Box* as a way of setting up the initial conditions of a new imagining. After that, the *Updater* mechanism is responsible of pulling the agent's beliefs into the scenario to complement whatever has not been specified by the initial premise, as well as inferring what else would follow from that initial setting. Finally, the authors call for a *Script Elaborator* mechanism as being responsible for coming up with new premises that would not naturally follow from the imagining and adding them there.

Furthermore, Williamson's recent work on the two modes of imagination in [25] shares a certain structural similarity with these two theories as well. The author also identifies what he calls a

[2] In relation to [22], we can say that our interest lies in *constructive imagining*.

voluntary mode of imagination in which the agent chooses and sets the initial conditions of an imagining, followed by an *involuntary* mode[3] in which the agent's knowledge and beliefs about the real world unfold what else would be the case there, given those initial conditions. Because Williamson's work is focused on these two mechanisms, he does not provide an account of those new voluntary additions that the former theories identified.

When focusing on the procedural account of imagination acts given by these authors, it can be seen how, even though they use different names and may have some minor differences, they all identify a similar structure[4] regarding how imaginary scenarios are created via a *voluntary initiation*, elaborated via a *reality-oriented development*, and new details added there via an *atypical development*. We summarize this structural similarity in Table 1, and we distill this shared structure to guide our dynamic formalization of imaginary scenarios throughout the rest of this work.

	Langland-Hassan	*Nichols / Stich*	*Williamson*
Voluntary initiation	Top-down intentions	Premises into PWB	Voluntary imagination
Reality-oriented dev.	Lateral constraints	UpDater	Involuntary imagination
Atypical development	Cyclical top-down intention	Script Elaborator	-

Table 1: A common underlying structure for the theories of imagination.

When considering formal systems, few authors have ventured into the uncharted seas of logic and imagination. David Lewis, in [15] defines a logic to account for counterfactual reasoning by using a system of spheres and a modal operator that moves the evaluation point to counterfactual worlds. Later, in [21], Niiniluoto formalizes imagination as a propositional attitude and discusses some of its properties. Costa-Leite, in [10], goes one step beyond and formalizes the distinction between "imagination", "conception" and "possibility" through following the intuitions of Descartes and Hume. Wansing brings beliefs into the picture in [24] and uses neighborhood semantics and STIT mechanics to account for agentive imagination. Through various works like [4], [5] and [6], Berto formalizes conceivability in both a paraconsistent and a classical setting, and introduces the mechanics of "aboutness", which determines what is relevant for the agent to import conceiving an alternative world.

Even though these works highlight very interesting features of imagination, they all represent imagination in *static*, pre-determined scenarios, like snapshots of a specific moment. Although Wansing's approach goes one step beyond and takes into account the agentive character of imagination, it still works in predefined, tree-like structures: the agent can be seen as "choosing" what to imagine, indeed, but these choices are already contained in the initial model of the situation.

Our approach amends this and captures something that has been overlooked in previous works: imagination is, in essence, dynamic. When we imagine, we create and unfold worlds that are not real, but which nevertheless are governed by a certain set of rules or mechanisms, as identified by the previously mentioned theories of imagination. Even though in [9] we define a formal system for the dynamics of imagination acts, our approach there focuses on dissecting a fine-grained account of what happens *within* an imagination act; in the present work, nevertheless, we consider an imagination act as a whole and represent it through a single step.

2.1 Distilling an Algorithm for Imagination

The main goal of the present work is to represent, using a dynamic formal system aided by an algorithm, how an agent creates new imaginary scenarios as a result of executing a voluntary act of imagination. Particularly, in this work we are not concerned with *what* results from an act of imagination (meaning its possible outcomes, or how such outcomes affect other mental attitudes such as beliefs), but rather with *how* this act of imagination is performed.

[3]Note that Williamson's theory distinguishes two modes involved in a voluntary act of imagination, in the sense we pointed out by the beginning of this section, and which is set and unfolded in two different phases. His use of the word "involuntary", nevertheless, still refers to a mechanism that belongs to an act of imagination that is willingly initiated by the agent.

[4]As we argue in [8], an alternative interpretation of the former theories could potentially lead to a different set of dynamic mechanisms. However, in the present work we choose to stick to the original interpretation of those theories.

In this first section, we discuss and sketch the intuitive mechanics of what we call the *Imagination Algorithm*[5]. This algorithm will then be captured in a formal way when defining the Logic of Imaginary Scenarios, and it will be the process governing how such imaginary scenarios are created and developed.

2.1.1 Creating a New Imaginary Scenario

When an agent decides to perform an act of imagination, the overall algorithm governing such decision must start by finding a spot within the mind[6] of the agent in where to create the imaginary scenario she is going to entertain.

The content of this new imaginary scenario must be defined, at the beginning, only by the initial premise used to create it. The reason why we require this to be the case follows from what Nichols and Stich define in their cognitive theory of pretense, as introduced in Section 2: the initial premise defining how an imaginary scenario should be has preference over anything else and, therefore, it is "clamped" into such scenario. This is done in order to be able to imagine anything[7] that may differ from our knowledge or beliefs: if we could not clamp the initial premise characterizing an imaginary scenario, then any premise contradicting our beliefs would be instantly overridden when merged with them.

Figure 1 represents what we require of this first mechanism: when executing an act of imagination, the first thing we must do is to create a new imaginary scenario in which we can clamp the initial premise φ that characterizes it.

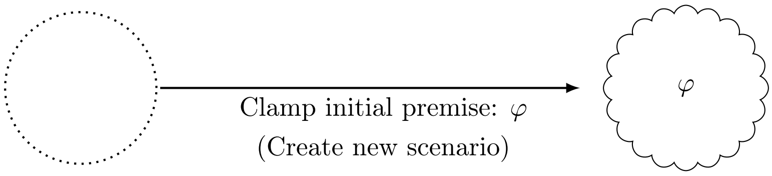

Figure 1: Creating a new imaginary scenario clamps the initial premise.

2.1.2 Importing Knowledge and Beliefs

Once the new imaginary scenario has been created, and the initial premise has been clamped in it, it is time to elaborate on the details of the scenario that were not determined by this initial premise. As we have seen when reviewing the theories of imagination in Section 2, the way imaginings develop follow the same reasoning mechanisms used for our beliefs. This phenomenon accounts for what is known as "reality-oriented development". When integrating this mechanism into the overall algorithm, we must look into the agent's knowledge and beliefs while taking into account, nevertheless, that the initial premise should still be considered clamped, and so it must have priority over what the agent knows or believes in the real world; otherwise, we would risk losing up, precisely, what makes the imaginary scenario different from reality. In order to do so, the agent should consider her knowledge and beliefs about the real world and, if they do *not* contradict the premise clamped into the imaginary scenario, then import them into it.

Figure 2 represents the way this mechanism works. Note how we draw this mechanism upon the previous Figure 1, in order to show how both mechanisms work together. In Figure 2, the circle on the left side contains what the agent knows and believes about the real world (in this case, $\neg\varphi$

[5]The term "algorithm" may sound controversial, when referring to acts of imagination. The formation of imaginary scenarios is, arguably, one of the most creative actions an agent can engage in and, as such, aiming to fully capture this process using an algorithm can seem reductive. Our algorithm does not aim to comprehensibly capture all that imagination can potentially offer, but rather to capture how an act of imagination can be formally represented in a possible worlds model through the addition of new worlds that follow, when determining what their content should be, a certain set of algorithmic rules Our use of the term "algorithm" should be interpreted as directly related and dependent on the logical language and the formal models behind.

[6]Similarly to [20] (in page 121), we want to stress the fact that we use expressions like "finding a spot", or "place within the mind" without implying, nor defending the existence of any kind of separate, specific physical place within the agent's mind.

[7]The phenomenon known as *imaginative resistance* states that there are certain things that we simply cannot imagine. Although we do not consider these cases here for the sake of simplicity, we refer to [18] for more on this topic.

and ψ, and possibly something else; note how, for the sake of simplicity, we omit the content of this world in Figure 1); note how, when importing the agent's knowledge and beliefs, as $\neg\varphi$ would contradict φ (which is the initial premise that was clamped into the imaginary scenario, and so it has preference), it is *not* imported.

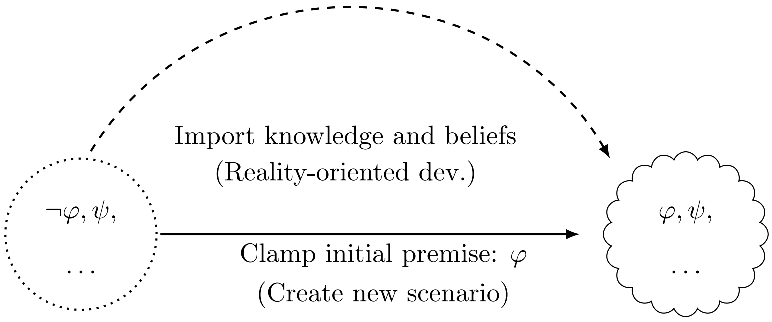

Figure 2: Import facts about the real world.

2.1.3 Adding New Premises

The last mechanism involved in an act of imagination is very similar to the first one; in fact, and as Langland-Hassan suggests in his theory (see Section 2), this voluntary addition of new premises into the imagining can be seen as a cyclical process that begins anew the whole cycle. We follow Langland-Hassan in this mechanism, and thus treat this addition of new premises as a sort of new imagining-initiating mechanism.

However, this addition of new premise differs from the initial creation of the imaginary scenario in an important way; namely, the initial premise used to characterize a new imaginary scenario was clamped on a blank, brand-new imagining, whereas the new premise that must be added in this process should be clamped into an imagining which has been already characterized in a previous step, and also elaborated through importing the agent's knowledge and beliefs. The main difference, in this case, is that the agent's knowledge and beliefs will be imported from the imaginary world where the agent adds a new premise, instead of going back to the real world. As the agent is elaborating on the details of an imagining that can already differ from the real world in some ways, her knowledge and beliefs must refer now to that imagining. Besides, note how, in this case, a new premise may be in conflict with some previous premise used to define the imaginary scenario. Nevertheless, and following Nichols and Stich's theory, as soon as the process of importing knowledge and beliefs is over, the initial premises lose their "privileged status" of being clamped, and so they can be overriden when new premises come into play.

Figure 3 includes this last mechanism upon the previous Figure 2, and thus represents the whole Imagination Algorithm. Concerning this last mechanism, note how the process of importing the agent's knowledge and beliefs is now based upon the previous imaginary world, rather than the real world that was used before.

2.1.4 Wrapping Up the Imagination Algorithm

The previous sections provide insights about the different mechanisms that take part in an execution of the Imagination Algorithm. Figure 3, specifically, represents the whole cycle of executing the Imagination Algorithm. In fact, it represents more than that, as the addition of a new premise into an already existing imaginary scenario begins the process anew, and so corresponds to a new call to the algorithm.

Therefore, and by taking into account how we understand the voluntary additions discussed in the previous section, we intuitively define an execution of the Imagination Algorithm as follows:

1. The execution requires an initial premise φ characterizing the initial scenario, and a *world of reference* upon which the agent bases her imagining.
2. The algorithm creates a new imaginary world and clamps φ into it.

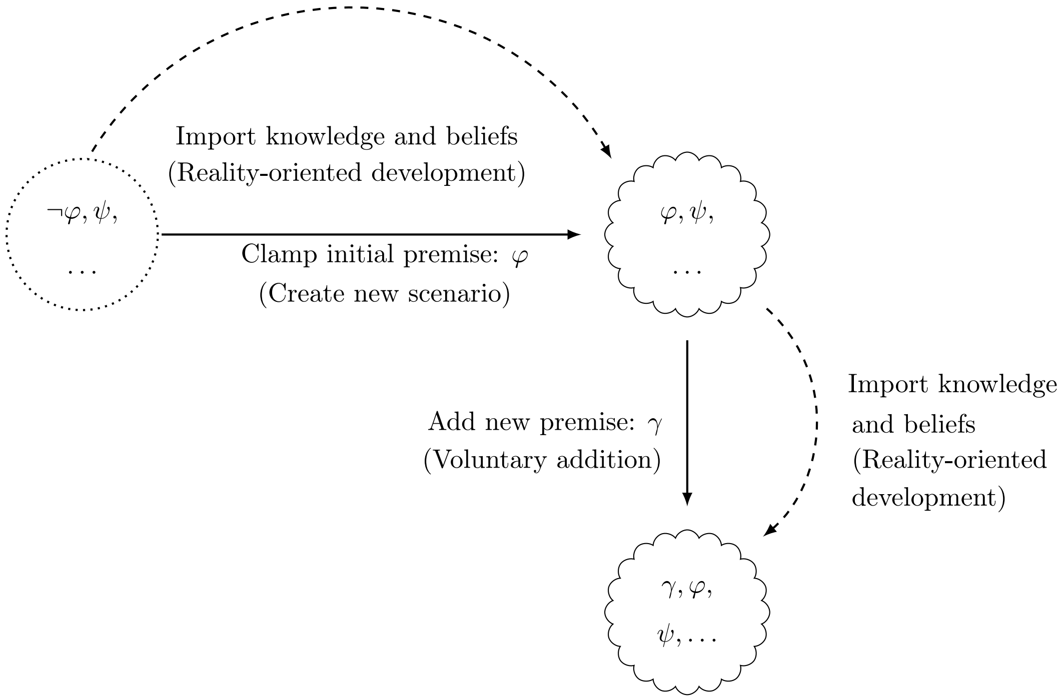

Figure 3: The full Imagination Algorithm.

3. The algorithm goes over what the agent knows or beliefs in the world of reference, and, if it does not contradict φ, imports it into the imaginary world.

4. The imaginary world is related to the world of reference through an act of imagination executed by the agent, with initial premise φ.

Note how, by introducing the notion of "world of reference", we already account for both those acts of imagination used to create a new imaginary world (and thus the ones that take the real world as the world of reference), but also for those acts of imagination used to add new premises into an already existing imaginary world (in which case, the world of reference is not the real world, but also an imaginary one).

Figure 4 highlights, using the same example as the one depicted in Figure 3, the two different acts of imagination that take place in there: the one used to create a new imaginary world from scratch by using an initial premise φ (and importing the agent's knowledge and beliefs from the real world), and the one used to add a new premise γ into an imaginary world that already exists (importing knowledge and beliefs from that imaginary world, in this case). For the sake of readability, we omit the labels explaining the relations.

3 The Logic of Imaginary Scenarios

In this section, we introduce all the formal details needed to define the Logic of Imaginary Scenarios, including a formal definition of the Imagination Algorithm, which defines how new imaginary worlds are created and elaborated in the formal models of our logic, and according to the insights gained from the previously reviewed theories of imagination.

3.1 Syntax

As imagination is related to other mental attitudes, we want to define our system upon a logic already able to handle, at least, some of those mental attitudes; however, we also want to build our proposal step by step, and without being overwhelmed by technical difficulties inherited from the background system used. Therefore, we build our proposal upon a *single-agent epistemic logic*

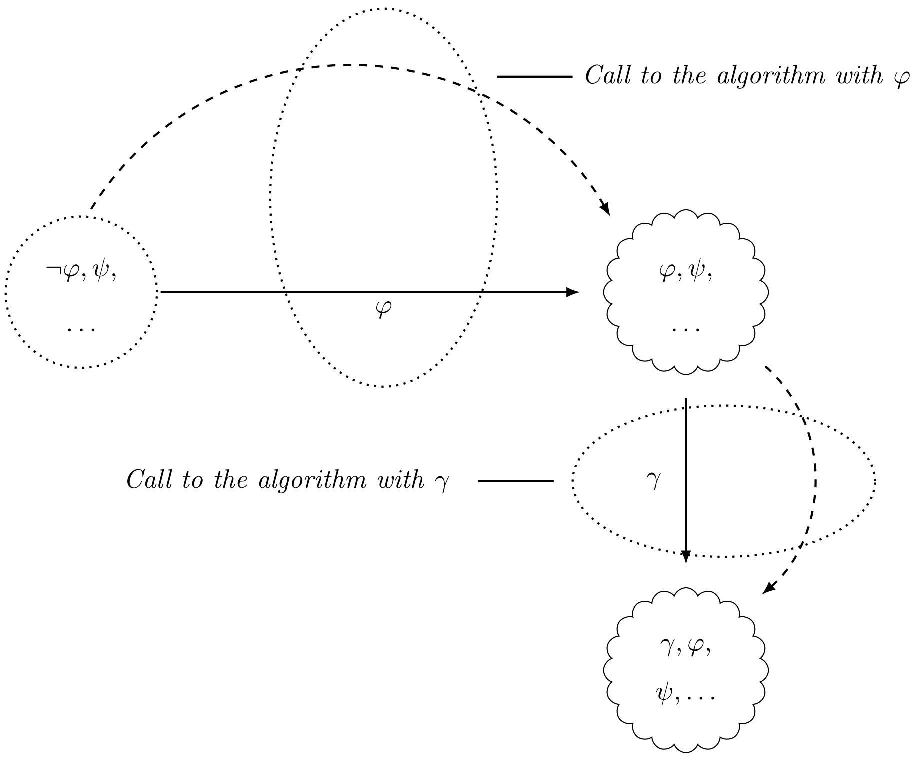

Figure 4: Two different executions of the Imagination Algorithm.

(see [17] for a comprehensive guide on epistemic and doxastic logics). Furthermore, we also add some features of *hybrid logic* (introduced in [7], for instance) into our initial mix.

It is worth noting that, while presenting the language and semantics of our logic, there will be some elements which we will need to mention before introducing: this is because the language, the models and the dynamic part of our proposal (handled by the Imagination Algorithm) are closely related between them. However, we will try to give an intuitive understanding of each notion before formally defining it.

The language of the *Logic of Imaginary Scenarios* is formed by a countably infinite set of *atomic propositions*, called ATOM, and represented by the lowercase letters $p, q, r \dots p_1, p_2 \dots$; besides, there is a countably infinite set of *nominals* (taken from hybrid logic), represented by the lowercase letters $i, j, k \dots i_1, i_2 \dots$ and called NOM.

We use the standard propositional connectives $\neg, \wedge, \vee, \rightarrow$ (standing for "negation", "conjunction", "disjunction" and "material implication", respectively); besides, we include the unary "knowledge" operator K, taken from epistemic logic, and the unary "at" operator @ taken from hybrid logic. Furthermore, we also introduce two new operators: a dynamic unary operator $\mathrm{Img}(\delta)$ called "dynamic imagination" and an unary modal operator $\langle I(\delta)\rangle$ called "static imagination"; both operators are signed with a formula δ of a special kind —which we introduce in the following lines.

We use bracket symbols (, [,),] as usual, and usually omit them when the context is clear. Now, the well-formed formulas of the language are defined recursively:

$$i \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid K\varphi \mid @_i\varphi \mid \mathrm{Img}(\delta) \mid \langle I(\delta)\rangle\varphi$$

where $i \in$ NOM, $p \in$ ATOM, $\{\varphi, \psi\} \subseteq$ FORM and $\delta \in$ FORM*. We recursively define the subset of formulas FORM$^* \subset$ FORM as follows:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi$$

where $p \in$ ATOM and φ either in ATOM or in FORM*. That is: FORM* is the propositional fragment of FORM. From now on, we use variables $\delta, \gamma, \dots$ to refer to elements of FORM*, in order to distinguish them from formulas belonging to FORM.

We also introduce two symbols $\top, \bot$ to refer to *truth* and *falsity*, respectively, and we define them as follows (for $p \in$ ATOM):

$$\top := p \vee \neg p$$
$$\bot := p \wedge \neg p$$

The intuitive, informal reading of $\text{Img}(\delta)$ would be "the agent creates an imaginary scenario using premise δ", whereas $\langle I(\delta)\rangle\varphi$ stands for "in an imaginary scenario initiated by premise δ, it is the case that φ".

Following the intuitions of the theories previously reviewed in Section 2, the dynamic operator $\text{Img}(\delta)$ is responsible for calling the Imagination Algorithm: in particular, it captures the fact that the agent decides to initiate an imaginary scenario characterized by an initial premise δ. We provide the formal definition of such algorithm in Section 3.3. Regarding the static operator $\langle I(\delta)\rangle$, note how it corresponds to a sort of static evaluation of an act of imagination that has *already* been performed: in this sense, this operator does not aim to represent any of the mechanisms involved when performing an act of imagination, but rather to evaluate an imaginary scenario, once it has already been created.

It is worth noting that, although at this stage we build our proposal upon an existing logic, we want to keep this underlying system as simple as possible: as a consequence, we do not have an explicit representation of *beliefs* in our logic. Therefore, we define a derived operator M to represent a weak form of belief, and we interpret it as being complementary to knowledge:

$$M\varphi := \neg K \neg \varphi$$

Intuitively, if the agent does *not* know $\neg\varphi$, then it is because she considers that φ could be the case as well: therefore, we could say that the agent believes φ (understanding this notion of "believing" as considering it possible to be the case, as far as the agent knows). Although this is a rather simplified account of beliefs, it will allow us to concentrate, at this stage, on capturing the dynamics of imagination acts.

3.2 The Models for Imaginary Scenarios

Similarly to what we do with the language of our logic, we build our models upon a standard model of single-agent epistemic logic, plus the elements introduced by hybrid logic. We take this model as basic, and we add a new accessibility relation upon it in order to account for imagination acts.

It is worth stressing the fact that, unlike most logic systems that represent static scenarios, our models are intended to represent the change involved in performing an act of imagination; therefore, they are *dynamic* by definition.

Typically, our models will be initially defined as being single-agent epistemic models, without any act of imagination represented in them yet. The interest of our proposal is, precisely, to allow for these acts of imagination to "happen" within our model, thus expanding it as a consequence. Therefore, we require every element of the relation R_{Img} (introduced in the following lines, and representing an act of imagination) to be created by explicitly following our Imagination Algorithm (formally defined in Section 3.3), thus ensuring that both the accessibility relation, and the imaginary worlds created by the algorithm, fulfill the conditions imposed by the way the algorithm behaves.

Definition 3.1. *A Model for Imaginary Scenarios is a structure $\mathcal{M} = \langle W, R_K, R_{Img}, V, N\rangle$ formed by the following elements (aside from R_{Img}, they are all taken from single-agent epistemic logic and hybrid logic):*

- *W is a non-empty set of elements called possible-worlds or states of affairs. We use the lowercase letters $w, v, u, \dots w_1, w_2, \dots$ to refer to elements of W. Among these possible worlds, we distinguish a special subset of "empty worlds" $W^{\emptyset} \subseteq W$, which are required to fulfill certain conditions stated after this definition, and which we require to have an infinite number of them.*

- *$R_K \subseteq W \times W$ is a binary relation over elements of W called the indistinguishability relation, and which we require to be reflexive, transitive and symmetric (due to restrictions typically imposed to knowledge: see [17], for instance). Intuitively, this relation establishes which possible worlds the agent thinks that can be the actual case, as far as she knows. We use pairs of the form $(w, v), (v, u), \dots$ to refer to elements of R_K.*

- $R_{Img} \subseteq W \times W \times FORM^*$ *is a ternary relation called the imagination relation. Intuitively, an element* (w, v, δ) *captures how, by performing an act of imagination "with content* δ*, and by taking* w *as the world of reference (in terms of being the possible world the agent considers to represent the actual case), an imaginary world* v *is created. We use triplets of the form* $(w, v, \delta), (u, z, \gamma), \ldots$ *to refer to elements of* R_{Img}*.*

- $V : ATOM \to \mathcal{P}(W)$ *is a function from atomic formulas of the language to subsets of the power set of* W*, called the valuation function. Intuitively, it keeps track of which atomic formulas are true at which subset of possible worlds.*

- $N : NOM \to W$ *is an exhaustive function setting, for each element of NOM, a unique possible world in* W*. Intuitively, this function sets which nominal is used to "name" each world.*

Among the set W of possible worlds, we have a subset $W^{\emptyset}$ of *empty worlds*, which are those possible worlds that do *not* appear in neither R_K, R_{Img} nor V; that is, those worlds $\{w \in W^{\emptyset} \mid (w, v) \notin R_K$ and $(v, w) \notin R_K$ and $(w, v, \delta) \notin R_{Img}$ and $(v, w, \delta) \notin R_{Img}$ and $w \notin V(p)\}$, for any world $v \in W$, any formula $\delta \in$ FORM* and any atom $p \in$ ATOM. Intuitively, empty worlds already exist in the model, but they do not represent any particular state of affairs, nor they are accessible through any accessibility relation. They do, however, have a certain nominal associated, even if they are not relevant at this point. As we will see in Section 3.3, these empty worlds will be used to represent new imaginary worlds after the agent performs an act of imagination.

By the way imaginary worlds are created by the Imagination Algorithm, a Model for Imaginary Scenarios represents different "clusters" of possible worlds; later on, after providing a formal definition of the algorithm in Section 3.3, we present a detailed example that shows how the system works.

3.3 The Imagination Algorithm

From now on, we use the term `ImgAlg` as a way of referring to the formal Imagination Algorithm defined within the Logic of Imaginary Scenarios. We have said that an execution of the algorithm requires an initial premise, and a world of reference: we will refer to the initial premise as δ (a formula in FORM*), and to the world of reference as w^R. Therefore, a call to the algorithm is expressed as follows:

$$\texttt{ImgAlg}(\delta, w^R)$$

We already know, at an intuitive level, what the role of δ is. When translating its role into the formal approach, δ is a formula that must hold (i.e.; it must be "clamped") at the imaginary world created by the corresponding imagination act. In other words, δ must be used to determine the atomic valuation of the worlds that will be created by the execution of $\texttt{ImgAlg}(\delta, w^R)$, as it is precisely the atomic valuation of the resulting world which will determine whether δ holds in there[8].

There is still one further notion that we need to introduce. During an execution of the `ImgAlg` upon the model for imaginary scenarios $\mathcal{M}$, the model gets expanded: this is, precisely, what we want to capture by understanding an act of imagination as an action that creates new imaginary worlds. Therefore, during the process of executing the `ImgAlg` upon a model $\mathcal{M}$ at the world of reference w^R, we end up having more possible worlds and more accessibility relations than we had just before executing the algorithm. In order to refer to the new model we introduce one further concept: the *expanded model* of $\mathcal{M}$, to which we refer to as $\mathcal{M}^+$. Thus, from now on, we may refer to either the whole expanded model $\mathcal{M}^+$, or to any of its elements as follows:

[8]Note that we restrict δ to belong to the propositional fragment of the language, and so we do not allow δ to include any kind of modal or hybrid expression. Regarding the restriction on modal operators, it is due to a lack of expressive power of our language. If we wanted to allow our agent to imagine that her knowledge is somehow different, our algorithm would need to create not just a new imaginary scenario, but rather a whole relational structure formed by different imaginary worlds aimed to represent how the agent imagines her knowledge would be. However, if we wanted to do so, we would need to be able to quantify over nominals, and we want to keep our initial setting simple: adding quantification involves a series of technical difficulties which we will leave for an extended version of the present system. Regarding the restriction imposed on hybrid operators, it is due to philosophical reasons. It does not seem to make sense for an agent to imagine something *about* a specific, already made and defined world. The agent must indeed take one world as the "world of reference" for her imaginings, but she cannot imagine that *that* specific world changes; rather, she must create a new imaginary world which, although being based on that one, will still be different.

$\mathcal{M}^+ = \langle W, R^+_K, R^+_{Img}, V^+, N\rangle$. We provide the formal definition of each of these expanded elements in the following paragraphs.

Now, the following steps define how the `ImgAlg` works, with respect to a Model for Imaginary Scenarios $\mathcal{M}$, a formula $\delta \in \text{FORM}^*$ and a possible world of reference $w^R \in W$:

1. The algorithm `ImgAlg` starts by being called with arguments δ and w^R. If formula δ is contradictory, the execution of the `ImgAlg` ends at this point[9].

2. In order to handle the formula in an efficient way, we compute the *Disjunctive Normal Form*[10] (DNF from now on) of δ, to which we refer as $\text{DNF}(\delta) := \delta_1 \vee \ldots \vee \delta_n$.

3. The `ImgAlg` must "create" a new imaginary world[11] for each possible alternative satisfying formula δ. Therefore, and recalling that $\text{DNF}(\delta) = \delta_1 \vee \ldots \vee \delta_n$, the `ImgAlg` must locate n empty worlds in $W^\emptyset$, to which we will refer as $w_1, \ldots, w_n$.

4. Once the new possible imaginary worlds have been selected, the `ImgAlg` must create new imaginary relations expressing that, when imagining formula δ at the world of reference w^R, the agent can access the new imaginary worlds $w_1, \ldots, w_n$. This defines the *expanded set of imaginary relations* as follows:

$$R^+_{Img} = R_{Img} \cup \Big(\bigcup_{i=1\ldots n} \{(w^R, w_i, \delta)\} \Big)$$

5. As any of the new imaginary possible worlds satisfies what the agent is imagining (specifically, δ), they should all be epistemic alternatives to the other imaginary worlds considered in this execution of the `ImgAlg`; in other words, the agent must consider them all as a possible way of representing an imaginary world satisfying δ. This defines the *expanded set of epistemic indistinguishability relations* as follows:

$$R^+_K = R_K \cup \Big(\bigcup_{\substack{i=1\ldots n \\ j=1\ldots n}} \{(w_i, w_j)\} \Big)$$

   As a consequence, $W^\emptyset$ is updated as $W^{\emptyset+} = W^\emptyset - \{w_1, \ldots, w_n\}$.

6. Let $k_1, \ldots, k_n$ the nominals corresponding to $w_1, \ldots, w_n$ with function N.

7. Last but not least, the `ImgAlg` must expand the valuation function to account for the atomic propositions holding at the new imaginary possible worlds. In order to do so, the algorithm must account for both the literals that appear in each δ_i, and also for the atoms that are true in the world of reference w^R and which should be imported to the new imaginary worlds, provided they do not appear in δ; this is so because any atom appearing in δ has preference over the atoms of the world of reference (the agent "clamps" the initial premise into the imaginary worlds). Therefore, the definition of the *expanded valuation function* involves two different phases:

   (a) Firstly, the `ImgAlg` must set the new valuation functions according to the atoms p appearing in δ_i, for each new imaginary possible world w_i:

$$V^+_1(p) = V(p) \cup \Big(\bigcup_i \{w_i \mid p \in PL(\delta_i)\} \Big)$$

   Where $PL(\delta_i)$ stand for the set of all the positive literals appearing in δ_i.

[9] As `ImgAlg` is executed on a model $\mathcal{M}$, in case δ is contradictory the algorithm does not expand $\mathcal{M}$ in any way; therefore, we can consider that, in that case, the algorithm returns $\mathcal{M}^+ = \mathcal{M}$.

[10] For a more comprehensive explanation of how the DNF of a formula can be computed, see [16]. For the present case, it suffices to say that every formula of propositional logic can be expressed in its equivalent DNF formula by following a simple algorithm.

[11] Note that, although we may informally keep referring to the creation of new imaginary worlds, a model $\mathcal{M}$ is specified in such a way that it already contains a countably infinite number of possible worlds W, among which we have a subset of countably infinite empty worlds $W^\emptyset$ (see Section 3.2); therefore, the `ImgAlg` does not actually "add" new possible worlds to the set W, but it just selects a specific number of empty possible worlds belonging to $W^\emptyset$ that are not "used" in model $\mathcal{M}$ by not being accessible nor having their atomic valuation specified, and uses them to define a new imaginary world that will then be plugged into the relevant part of the model through the corresponding accessibility relations. Recall that those empty worlds are already associated with their corresponding nominal through function N, so the `ImgAlg` does not have to modify neither the set of nominals NOM, nor the function N associating them to the imaginary worlds.

(b) Then, it must import all the atoms that are true at the world of reference w^R, provided they do not appear in δ_i, for each new imaginary possible world w_i:

$$V^+(p) = V_1^+(p) \cup \Big(\bigcup_i \{w_i \mid w^R \in V_1(p) \text{ and } p \notin NL(\delta_i)\}\Big)$$

Where $NL(\delta_i)$ stand for the set of all the negative literals appearing in δ_i.

8. The `ImgAlg` has finished its execution: a new set of imaginary possible worlds satisfying δ has been defined, these worlds are now accessible through the imagination relation R^+_{Img} from the world of reference w^R, and they are epistemically indistinguishable by the agent in the corresponding imaginary scenario.

The Imagination Algorithm has been formally defined as the `ImgAlg`, and it can now be executed to expand a model $\mathcal{M}$ into a model $\mathcal{M}^+$, which includes a set of new imaginary possible worlds that were not accessible before, and that result from the agent performing an act of imagination with an initial premise δ. In brief, what the `ImgAlg` does is to select a subset of empty worlds $\{w_1, \ldots, w_n\} \subset W^\emptyset$, which were not accessible before, nor had any atomic valuation specified, and makes them accessible through adding new relations in R^+_{Img} and R^+_K as required; besides, the `ImgAlg` adds these new worlds to the valuation function V as corresponds. Note that, once these worlds $w_1, \ldots, w_n$ have been added to the accessibility relations and the valuation function, they will not longer belong to the subset of empty worlds $W^\emptyset$, as they would now be a "visible" part of the expanded model.

As it can be seen in the previous specification of the `ImgAlg`, the only restriction we put on the content of the act of imagination is that it can be expressed in the propositional fragment of our logic (that is, we require it to belong to FORM*), and we do not allow our agent to imagine contradictory premises. Aside from that, `ImgAlg` provides the required mechanisms to allow the agent to imagine whatever she wants to, and expands the model in consequence by adding new imaginary possible worlds.

An Example of an Act of Imagination

In the present section, we provide a brief example of how an execution of the `ImgAlg` works, given an initial Model for Imaginary Scenarios $\mathcal{M}$. Before any execution of the `ImgAlg` (or, in other words, before performing any act of imagination), a Model for Imaginary Scenarios looks like a standard, single-agent epistemic model (in particular, the imagination relation R_{Img} is empty, as no act of imagination is represented in there yet), like the one in Figure 5.

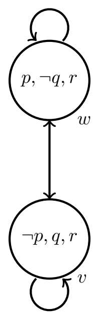

Figure 5: The initial model $\mathcal{M}$, before performing any act of imagination.

Now, our agent decides to perform an act of imagination with an initial premise $\delta = (\neg p \vee \neg r) \rightarrow ((q \wedge \neg r) \vee \neg q)$. Note that, although one of the existing possible worlds already satisfies this formula (specifically world v, by making p false in it), performing such act of imagination will create possibly many different possible worlds in which the formula $(\neg p \vee \neg r) \rightarrow ((q \wedge \neg r) \vee \neg q)$ holds, even if they are not already considered as epistemically possible by the agent.

In the following lines we briefly go over what the `ImgAlg` does when computing the expanded model $\mathcal{M}^+$, which is the expanded version of $\mathcal{M}$ in which there are some new imaginary possible worlds that are now accessible through relation R_{Img}, and which satisfy the initial premise δ used by the agent to imagine those worlds. The final model can be seen in Figure 6, and this model is computed as follows:

1. To begin with, the `ImgAlg` checks whether $\delta = (\neg p \lor \neg r) \to ((q \land \neg r) \lor \neg q)$ is contradictory; as it is not, the execution keeps moving forward.

2. The DNF of δ is $(p \land r) \lor (q \land \neg r) \lor \neg q$; as there are 3 different clauses in this DNF, the `ImgAlg` will look for 3 empty worlds $\{w_1, w_2, w_3\} \subset W^{\emptyset}$, which will represent the new imaginary worlds created as a result of the current act of imagination.

3. The `ImgAlg` must make these new worlds w_1, w_2 and w_3 accessible from the world of reference w, through imagining $\delta = (\neg p \lor \neg r) \to ((q \land \neg r) \lor \neg q)$ and by expanding relation R_{Img} into R^{+}_{Img} by adding those new relations.

4. Now, as each new imaginary world w_1, w_2 and w_3 accounts for a different worlds where δ holds, we consider them to be epistemically indistinguishable by the agent, in terms that each one of them satisfies what the agent wants to imagine. In order to do so, we must expand relation R_K into R^{+}_{K} by adding the required relations to make the relations between worlds w_1, w_2 and w_3 reflexive, symmetric and transitive, as required in this relation.

5. The `ImgAlg` must determine the atomic propositions holding at each new imaginary world. This is done in two different steps:

   (a) Firstly, the algorithm must check each clause δ_i in DNF(δ) and determine the value of the atoms appearing in δ_i, for each corresponding new imaginary world w_i. That is, in this example the algorithm must check $(p \land r)$ for world w_1, then $(q \land \neg r)$ for world w_2, and finally $\neg q$ for world w_3. This determines which atomic propositions must be clamped at each new world.

   (b) Then, the `ImgAlg` must check those atomic propositions holding at the world of reference w and, for each new imaginary world w_i, import those ones that do not appear in the corresponding clause δ_i, as those would have priority over the new imported ones. In the current example, the algorithm must import p for world w_2 and also p for world w_3. Note how negative atoms are implicitly "imported", when required, by being automatically false in the imaginary worlds where their positive version has not been clamped before.

7. The `ImgAlg` has finished its execution. Figure 6 shows the expanded model $\mathcal{M}^{+}$ that results from executing an act of imagination with an initial premise $\delta = (\neg p \lor \neg r) \to ((q \land \neg r) \lor \neg q)$ at a world of reference $w^R = w$ in a Model for Imaginary Scenarios $\mathcal{M}$. For the sake of clarity, we highlight the clamped formulas in bold font, and we also represent the negated atomic formulas.

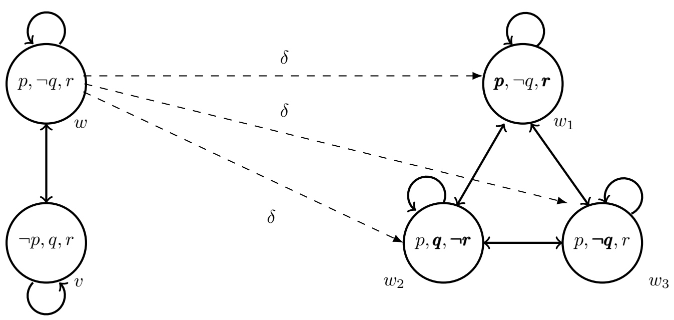

Figure 6: The expanded model $\mathcal{M}^{+}$ after an act of imagination.

3.4 Semantics

We evaluate a formula φ of the Logic of Imaginary Scenarios at a world $w \in W$ of a model $\mathcal{M}$ as follows:

Definition 3.2. *We use symbol $\vDash$, which we call local consequence, and we write $\mathcal{M}, w \vDash \varphi$ to express that φ is true at w in model $\mathcal{M}$; conversely, we write $\mathcal{M}, w \nvDash \varphi$ to express that φ is not true at w in model $\mathcal{M}$. Furthermore, we write $\vDash \varphi$ to express that φ is true at every world of every model; i.e., φ is a validity. Besides, we write $\Gamma \vDash \varphi$ (for Γ being a set of formulas) if, for every model $\mathcal{M}$ and world w such that $\mathcal{M}, w \vDash \Gamma$ (that is: every formula in Γ is true at world w of $\mathcal{M}$), it is the case that $\mathcal{M}, w \vDash \varphi$. In other words: any model satisfying the set of formulas Γ would also satisfy formula φ. In this case, we say that φ is a semantic consequence of Γ.*

We define $\mathcal{M}, w \vDash \varphi$ by induction on the formation of the formula φ as follows (where "iff" stands for "if and only if"):

1. $\mathcal{M}, w \vDash i$ iff $N(i) = w$ and, for every $v \in W$, if $\mathcal{M}, v \vDash i$, then $v = w$
2. $\mathcal{M}, w \vDash p$ iff $w \in V(p)$
3. $\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$
4. $\mathcal{M}, w \vDash \varphi \land \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$
5. $\mathcal{M}, w \vDash \varphi \lor \psi$ iff $\mathcal{M}, w \vDash \varphi$ or $\mathcal{M}, w \vDash \psi$
6. $\mathcal{M}, w \vDash \varphi \to \psi$ iff $\mathcal{M}, w \vDash \neg\varphi$ or $\mathcal{M}, w \vDash \psi$
7. $\mathcal{M}, w \vDash K\varphi$ iff for every world $v \in W$ such that $(w, v) \in R_K$, it is the case that $\mathcal{M}, v \vDash \varphi$
8. $\mathcal{M}, w \vDash @_i\varphi$ iff there exists a world $v \in W$ such that $N(i) = v$ and $\mathcal{M}, v \vDash \varphi$
9. $\mathcal{M}, w \vDash \mathrm{Img}(\delta)$ iff δ is not contradictory, $\mathrm{DNF}(\delta) := (\delta_1 \lor \ldots \lor \delta_m)$, $\texttt{ImgAlg}(\delta, w)$ has been executed and there are $w_1, \ldots, w_m \in W$ different from w and such that, for $l = 1, \ldots, m$ and $r = 1, \ldots, m$, the following holds:
   - $(w, w_l, \delta) \in R_{Img}$
   - $(w_l, w_r) \in R_K$
   - $w_l \in V(p)$ for all $p \in PL(\delta_l)$
   - if $w \in V(p)$ then $w_l \in V(p)$ for all $p \notin NL(\delta_l)$
10. $\mathcal{M}, w \vDash \langle I(\delta)\rangle\varphi$ iff $\mathcal{M}, w \vDash \mathrm{Img}(\delta)$ and there is some $v \in W$ such that $(w, v, \delta) \in R_{Img}$ and $\mathcal{M}, v \vDash \varphi$

It is worth devoting a few lines to clarifying how the dynamic operator $\mathrm{Img}(\delta)$ works. As we have already explained, this operator has the particularity of representing a voluntary action, performed by the agent, to imagine something (δ, specifically). The aim of this operator, therefore, is to validate the call of the `ImgAlg` with parameters δ and w (being w the world where the formula is evaluated and, thus, the world the agent takes as the reference to carry out such act of imagination). Providing satisfiability conditions for this operator, then, is intended to state the execution of a procedure that expands the model into its expanded version.

4 Soundness and Completeness

In the previous sections we have defined the language, semantics and the algorithm responsible for taking care of the way acts of imagination behave, when based on a single-agent epistemic setting. In this section, we provide a calculus for the Logic of Imaginary Scenarios, and we prove its soundness and completeness. Before digging deeper into the calculus, we define the notions of consequence following the standard approach:

4.1 Rules and Axioms

The following list contains the minimum set of rules and axioms needed for the calculus. Rules and axioms referring exclusively to the K operator are the usual ones and capture the properties of the relation R_K being reflexive, transitive and symmetric. Similarly, rules and axioms concerning the @ operator are the usual in hybrid logic (see [7], for example). There are two rules and three axioms specifically added for our system.

Rules

1. **Modus Ponens**: If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.
2. Hybrid and Epistemic Rules
   (a) **Gen$_@$**: If $\vdash \varphi$, then $\vdash @_i\varphi$.
   (b) **Gen$_K$**: If $\vdash \varphi$, then $\vdash K\varphi$.
   (c) **Name**: If $\vdash @_i\varphi$ and i does not occur in φ, then $\vdash \varphi$.
   (d) **Paste$_M$** : If $\vdash (@_iMj \wedge @_j\varphi) \to \psi$ and $j \neq i$ does not occur in φ and ψ, then $\vdash @_iM\varphi \to \psi$
3. Imagination Rules
   (a) **Paste$_{Img}$** : If $\delta \in FORM^*$, with DNF$(\delta) = \delta_1 \vee \ldots \vee \delta_m$ and $k_1, \ldots, k_m$ nominals different from i and not in δ or ψ:
   $\vdash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta)\rangle k_l) \wedge (\bigwedge_{l=1\ldots m} (\bigwedge_{p\in PL(\delta_l)} @_{k_l}p)) \wedge (\bigwedge_{l=1\ldots m} (\bigwedge_{p\notin NL(\delta_l)} (@_ip \to @_{k_l}p))) \to \psi$
   implies $\vdash @_i\mathrm{Img}(\delta) \to \psi$
   (b) **Paste$_I$** : If $\delta \in FORM^*$, with DNF$(\delta) = \delta_1 \vee \ldots \vee \delta_m$ and $k_1, \ldots, k_m$ nominals different from i and not in δ, φ or ψ:
   $\vdash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta)\rangle k_l) \wedge (@_{k_1}\varphi \vee \ldots \vee @_{k_m}\varphi) \to \psi$ implies $\vdash @_i\langle I(\delta)\rangle\varphi \to \psi$

Axioms

1. **Classical Tautologies**
2. Epistemic Axioms
   (a) **K$_K$**: $\vdash K(\varphi \to \psi) \to (K\varphi \to K\psi)$
   (b) **Reflexivity**: $\vdash K\varphi \to \varphi$
   (c) **Transitivity**: $\vdash K\varphi \to KK\varphi$
   (d) **Symmetry**: $\vdash \varphi \to KM\varphi$
3. Hybrid Axioms
   (a) **K$_@$**: $@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$
   (b) **Selfdual**: $\vdash @_i\varphi \leftrightarrow \neg @_i\neg\varphi$
   (c) **Ref**: $\vdash @_ii$
   (d) **Agree**: $\vdash @_i@_j\varphi \leftrightarrow @_j\varphi$
   (e) **Intro**: $\vdash i \to (\varphi \leftrightarrow @_i\varphi)$
   (f) **Back**: $\vdash M@_i\varphi \to @_i\varphi$
4. Imagination Axioms. If $\delta \in FORM^*$:
   (a) **Imaginary Possibilities**: $\vdash \langle I(\delta)\rangle i \wedge \langle I(\delta)\rangle j \to @_iMj$
   (b) **Imagination Bridge**: $\vdash \langle I(\delta)\rangle i \wedge @_i\varphi \to \langle I(\delta)\rangle\varphi$
   (c) **Voluntary Imagination** $\vdash \langle I(\delta)\rangle\varphi \to \mathrm{Img}(\delta)$

We introduce the usual concepts and definitions regarding deductions, and we end up by proving that the axioms of our system are sound.

Definition 4.1. *A deduction of φ is a finite sequence $\epsilon_1, \cdots, \epsilon_n$ of well-formed formulas such that, for every $1 \leq i \leq n-1$, either ϵ_i is an axiom or it is obtained from a previous formula in the sequence using the calculus rules, and $\epsilon_n := \varphi$.*

If $\Gamma \cup \{\varphi\}$ is a set of well-formed formulas of the language, a deduction of φ from Γ, written $\Gamma \vdash \varphi$, is a deduction $\gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$, where, for every $1 \leq i \leq n$, $\gamma_i \in \Gamma$.

We write $\vdash \varphi$ whenever we have $\emptyset \vdash \varphi$, and we will say that φ is a theorem of the calculus.

Theorem 4.2 (Soundness). *Let φ and Γ be (a set of) well-formed formulas of the language. The following statement holds:*

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$$

Proof. Correctness of epistemic rules is proved as usual in epistemic logic; similarly, soundness of hybrid rules and axioms is proved as usual in hybrid logic, the same for rule Modus Ponens, which comes from classical logic. Therefore, in the following lines we only provide a sketch of the proofs for rules 3a and 3b, and axioms 4a, 4b and 4c.

1. Rule Paste$_{Img}$ (3a). Take an arbitrary model $\mathcal{M}$, an arbitrary world $w \in W$, and suppose that δ is not contradictory, with DNF$(\delta) := (\delta_1 \vee \ldots \vee \delta_m)$ and $\mathcal{M}, w \vDash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta)\rangle k_l) \wedge (\bigwedge_{l=1\ldots m} (\bigwedge_{p \in PL(\delta_l)} @_{k_l} p)) \wedge (\bigwedge_{l=1\ldots m} (\bigwedge_{p \notin NL(\delta_l)} (@_i p \to @_{k_l} p))) \to \psi$ for $k_1, \ldots, k_m$ nominals different from i and not in δ or ψ. Now, suppose that $\mathcal{M}, w \vDash @_i \mathrm{Img}(\delta)$. Then $\mathcal{M}, v \vDash \mathrm{Img}(\delta)$ for $v = N(i)$. By the definition of satisfiability and the fact that $\texttt{ImgAlg}(\delta, v)$ has been executed we can prove that $\mathcal{M}, w \vDash \psi$.

2. Rule Paste$_I$ (3b). Take an arbitrary model $\mathcal{M}$, an arbitrary world $w \in W$, and suppose that δ is not contradictory, with DNF$(\delta) := (\delta_1 \vee \ldots \vee \delta_m)$ and $\mathcal{M}, w \vDash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta)\rangle k_l) \wedge (@_{k_1}\varphi \vee \ldots \vee @_{k_m}\varphi) \to \psi$, for $k_1, \ldots, k_m$ nominals different from i and not in δ or ψ. Suppose, also, that $\mathcal{M}, w \vDash @_i\langle I(\delta)\rangle\varphi$. Then $\mathcal{M}, u \vDash \langle I(\delta)\rangle\varphi$ for $u = N(i)$. This means that $\mathcal{M}, u \vDash \mathrm{Img}(\delta)$ and we know that $\texttt{ImgAlg}(\delta, u)$ has been executed. By using the definition of satisfiability we obtain $\mathcal{M}, u \vDash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta)\rangle j_l)$ and $\mathcal{M}, u \vDash (@_{k_1}\varphi \vee \ldots \vee @_{k_m}\varphi)$. And, thus, we obtain $\mathcal{M}, w \vDash \psi$, as we wanted to prove.

3. Axiom Imaginary Possibilities (4a). Take an arbitrary model $\mathcal{M}$, an arbitrary world $w \in W$, and suppose $\mathcal{M}, w \vDash \langle I(\delta)\rangle i$ and $\mathcal{M}, w \vDash \langle I(\delta)\rangle j$. By the definition of interpretation we have that $\mathcal{M}, w \vDash \mathrm{Img}(\delta)$ and there is $v = N(i)$ and $u = N(j)$ such that $(w, v, \delta) \in R_{Img}$ and $(w, u, \delta) \in R_{Img}$, and v and u have been created by execution of $\texttt{ImgAlg}(\delta, w)$, and thus $(v, u) \in R_K$, wich means that $\mathcal{M}, w \vDash @_i Mj$.

4. Axiom Imagination Bridge (4b). Take an arbitrary model $\mathcal{M}$, an arbitrary world $w \in W$, and suppose $\mathcal{M}, w \vDash \langle I(\delta)\rangle i \wedge @_i\varphi$. Then, $\mathcal{M}, w \vDash \langle I(\delta)\rangle i$ and $\mathcal{M}, w \vDash @_i\varphi$. Now, there is a world v such that $N(i) = v$ such that $(w, v, \delta) \in R_{Img}$ and $\mathcal{M}, v \vDash \varphi$. Thus, $\mathcal{M}, w \vDash \langle I(\delta)\rangle\varphi$.

5. Axiom Voluntary Imagination (4c). Take an arbitrary model $\mathcal{M}$, an arbitrary world $w \in W$, and suppose $\mathcal{M}, w \vDash \langle I(\delta)\rangle\varphi$. By definition of interpretation, $\mathcal{M}, w \vDash \mathrm{Img}(\delta)$.

□

In order to prove certain properties regarding the completeness of our system, we also need certain theorems of the calculus. Particularly, in the next pages we will be using the following theorems, which are taken from [1]. The proofs for these theorems also follow the ones in that work. The demonstrations are the usual ones in the propositional, epistemic and hybrid logics and we will omit them here.

Theorem 4.3. *The following holds.*

1. ***Deduction theorem:*** *If $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$.*
2. $\vdash @_i j \to @_j i$
3. $\vdash @_i j \to (@_i\varphi \leftrightarrow @_j\varphi)$
4. $\vdash @_i j \to (@_j k \to @_i k)$
5. $\vdash @_i Mj \wedge @_j\varphi \to @_i M\varphi$

4.2 Maximal consistency and saturation

In order to prove completeness of the logic, we first need to provide certain well-known definitions and theorems.

Definition 4.4. *Let Γ be a set of well-formed formulas:*

- *Γ is contradictory if and only if $\Gamma \vdash \bot$.*
- *Γ is consistent if and only if $\Gamma$is not contradictory.*
- *Γ is a maximal consistent set if and only if Γ is consistent, and whenever φ is a formula such that $\varphi \notin \Gamma$, then $\Gamma \cup \{\varphi\}$ is contradictory.*

Theorem 4.5 (**Consistency**)**.** *Let Γ be a set of well-formed formulas. The following holds:*

- *If Γ is consistent and $\Delta \subseteq \Gamma$, then Δ is consistent.*
- *If Γ is contradictory and $\Gamma \subseteq \Delta$, then Δ is contradictory.*
- *Γ is consistent if and only if every finite subset of Γ is consistent.*

Theorem 4.6 (**Maximal Consistency**)**.** *Let Γ be a maximally consistent set of formulas, and let φ, ψ be formulas. The following holds:*

- *$\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$.*
- *If $\vdash \varphi$, then $\varphi \in \Gamma$.*
- *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.*
- *$\varphi \wedge \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ and $\psi \in \Gamma$.*
- *Either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both.*

The following definition contains new concepts that will be fundamental in the completeness proof.

Definition 4.7 (**Saturation**)**.** *Let Γ be a set of well-formed formulas:*

1. *Γ is named if and only if at least one of its elements is a nominal $i \in NOM$.*
2. *Γ is M-saturated if and only if for all expressions $@_i M\varphi \in \Gamma$ there is a nominal $k \in NOM$ such that $@_k Mj \in \Gamma$ and $@_k\varphi \in \Gamma$.*
3. *Γ is Img-saturated if and only if for all expressions $@_i Img(\delta) \in \Gamma$ ($DNF(\delta) = \delta_1 \vee \ldots \vee \delta_n$) there are new nominals $k_1, \ldots, k_n \in NOM$ such that for all $l = 1, \ldots, n$ and $r = 1, \ldots, n$: $@_i\langle I(\delta)\rangle k_l \in \Gamma$, $@_{k_l} p \in \Gamma$ for $p \in PL(\delta_l)$, and $@_{k_l} p \in \Gamma$ if $@_i p \in \Gamma$ for $p \notin NL(\delta_l)$.*
4. *Γ is I-saturated if and only if for all expressions $@_i\langle I(\delta)\rangle\varphi \in \Gamma$ then $@_i Img(\delta) \in \Gamma$ and there is a nominal $k \in NOM$ such that $@_i\langle I(\delta)\rangle k \in \Gamma$ and $@_k\varphi \in \Gamma$.*

We say that Γ is fully-saturated when it is M-, Img- and I-saturated.

4.3 Completeness

Theorem 4.8 (**Completeness**)**.** *Let φ and Γ be (a set of) well-formed formulas of the language. The following claim holds:*

$$\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$$

Proof. The proof follows the well known Henkin-style proof of completeness It is a collorary of the Henkin Theorem (4.9). □

Theorem 4.9 (**Henkin Theorem**)**.** *Let Γ be a set of well-formed formulas of the language. If Γ is consistent, then Γ has a model.*

Proof. The proof is standard, and it is a corollary of the Lindenbaum Lemma (4.10) and the Truth Lemma (4.19) □

Theorem 4.10 (**Lindenbaum Lemma**). *If Γ is any consistent set of formulas, there is an extension to a maximal consistent set Γ^* which is named, M-saturated, Img-saturated and I-saturated.*

Proof. The first thing to do is to construct the set Γ^*.

Definition 4.11. *Let $\{k_n\}_{n\in\omega}$ be an enumeration of a countably infinite set of new nominals and $\{\varphi_n\}_{n\in\omega}$ be an enumeration of all the well-formed formulas of the extended language. We require the enumeration of formulas to be sorted in such a way that every appearance of a formula $@_i Img(\delta)$ must appear just before any appearance of any other formulas of the form $@_i\langle I(\delta)\rangle\psi$ (for the same i and δ); moreover, we also require that every formula $@_i Img(\delta)$ must appear* after *any other formula of the kind $@_i\varphi$, for φ not of the form $\langle I(\gamma)\rangle\psi$ (for any $\gamma \in FORM^*$ and $\psi \in FORM$).*

We shall define by induction a family $\{\Gamma^n\}_{n\in\omega}$ of sets:

- $\Gamma^0 = \Gamma \cup \{i_0\}$*, with i_0 the first new nominal.*
- *Now assume that Γ^n has been defined; to define Γ^{n+1} we distinguish five cases:*
  1. *If $\Gamma^n \cup \{\varphi_n\}$ is inconsistent, then $\Gamma^{n+1} = \Gamma^n$.*
  2. *If $\Gamma^n \cup \{\varphi_n\}$ is consistent and φ_n is not of the form $@_i M\psi$, $@_i Img(\delta)$ or $@_i\langle I(\delta)\rangle\psi$, then $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\}$.*
  3. *If $\Gamma^n \cup \{\varphi_n\}$ is consistent and $\varphi_n := @_i M\psi$, then $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n, @_i Mk, @_k\psi\}$, with k the first new nominal not in Γ^n nor φ_n.*
  4. *If $\Gamma^n \cup \{\varphi_n\}$ is consistent and $\varphi_n := @_i Img(\delta)$ (with $\delta \not\equiv \bot$, and being DNF(δ) $= \delta_1 \vee \ldots \vee \delta_m$), then $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup (\bigcup_{l=1\ldots m} \{@_i\langle I(\delta)\rangle k_l\}) \cup (\bigcup_{\substack{l=1\ldots m\\ r=1\ldots m}} \{@_{k_l} Mk_r\}) \cup (\bigcup_{l=1\ldots m} \{@_{k_l} p \mid p \in PL(\delta_l)\})) \cup (\bigcup_{l=1\ldots m} \{@_i p \to @_{k_l} p \mid p \notin NL(\delta_l)\})$, with $k_1, \ldots, k_m$ being the first new nominals not in Γ^n nor in φ_n.*
  5. *If $\Gamma^n \cup \{\varphi_n\}$ is consistent and $\varphi_n := @_i\langle I(\delta)\rangle\psi$ (and $\delta \not\equiv \bot$), then $@_i Img(\delta) \in \Gamma^r$ for $r \leq n$ (because, due to the ordering restriction we impose on the enumeration of formulas, it must appear before φ_n, in step r) and $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{@_{k_1}\psi \vee \ldots \vee @_{k_m}\psi\}$, where $k_1, \ldots, k_m$ are the new nominals added in step r as a result of adding $\varphi_r := @_i Img(\delta)$.*
- *Let* $\Gamma^* = \bigcup_{n\in\omega} \Gamma^n$

Now we will prove tha Γ^* is a maximal consistent set, named, M-saturated, *Img*-saturated and I-saturated. We will proof this in three steps:

1. For every $n \in \omega$, Γ^n is consistent. The proof is by induction

   - $\Gamma^0 = \Gamma \cup \{i_0\}$ is consistent, because if we suppose that $\Gamma \cup \{i_0\}$ is inconsistent, then so $\Gamma \cup \{i_0\} \vdash \bot$. By using the deduction theorem and the Name rule we obtain a contradiction.
   - Assume Γ^n is consistent. Γ^{n+1} has to be of one of the following forms:

     (i) $\Gamma^{n+1} = \Gamma^n$. In this case it is consistent by the induction hypothesis.

     (ii) $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\}$. In this case it is consistent by construction.

     (iii) $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n, @_i Mk, @_k\psi\}$, where $\varphi_n := @_i M\psi$ and k the first new nominal not in Γ^n nor φ_n. We also know that $\Gamma^n \cup \{\varphi_n\}$ is consistent. Γ^{n+1} must be consistent, because if we suppose that it is inconsistent, by using the deduction theorem and Paste$_M$ rule (2d) we obtain a contradiction.

     (iv) $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup (\bigcup_{l=1\ldots m} \{@_i\langle I(\delta)\rangle k_l\}) \cup (\bigcup_{\substack{l=1\ldots m\\ r=1\ldots m}} \{@_{k_l} Mk_r\}) \cup (\bigcup_{l=1\ldots m} \{@_{k_l} p \mid p \in PL(\delta_l)\})) \cup (\bigcup_{l=1\ldots m} \{@_i p \to @_{k_l} p \mid p \notin NL(\delta_l)\})$ (with DNF(δ) $= \delta_1 \vee \ldots \vee \delta_m$), where $\varphi_n := @_i$Img(δ) and $k_1, \ldots, k_m$ are the first new nominals not in Γ^n nor in φ_n. We also know that $\Gamma^n \cup \{\varphi_n\}$ is consistent. Γ^{n+1} must be consistent, because if we suppose that it is inconsistent, by using the deduction theorem and rule 3a we obtain a contradiction.

(v) $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n, @_{k_1}\psi \vee \ldots \vee @_{k_m}\psi\}$, where $\varphi_n := @_i\langle I(\delta)\rangle\psi$, δ is not contradictory, and $k_1, \ldots, k_m$ are the new nominals added in step r (with $@_i\text{Img}(\delta) \in \Gamma^r$ for $r \leq n$). We have added the $k_1, ..., k_m$ and the corresponding formulas in step r, and, in particular $\bigcup_{l=1\ldots m} \{@_i\langle I(\delta)\rangle k_l\} \subset \Gamma^n$. Γ^{n+1} is consistent because if we suppose that it is inconsistent, by using the deduction theorem and rule number 3b we will obtain a contradiction.

2. Γ^* is consistent. The proof follows the standard approach: by supposing that it is inconsistent, there will be an inconsistent subset of Γ^*, which is a contradition by the way Γ^* has been built.

3. Γ^* is named, M-saturated, *Img*-saturated and I-saturated by construction.

□

Now, in order to obtain the Truth Lemma (4.19), we need to build a *canonical model* for Δ, a maximal consistent set of formulas which is named, M-saturated, *Img*-saturated and I-saturated. We start by defining an equivalence relation over the set of nominals NOM as follows:

Definition 4.12. *Let Δ be a maximal consistent, named and fully-saturated set. We define, for all $i, j \in NOM$, a binary relation $\sim_\Delta$ as follows:*

$$i \sim_\Delta j \textit{ if and only if } @_i j \in \Delta$$

Proposition 4.13. *The relation $\sim_\Delta$ is an equivalence relation over NOM.*

Proof. The proof is a consequence of Δ being a maximal consistent set, axiom 3c and theorem 4.3. □

We now define the equivalence class $[i] = \{j \in \text{NOM} \mid i \sim_\Delta j\}$. We use this equivalence class to define two different accessibility relations as follows:

Definition 4.14. *Let Δ be a maximal consistent, named and fully-saturated set, and let $[i]$ be the equivalence class of $i \in NOM$. We define an epistemic indistinguishability relation R_K^Δ as:*

$$R_K^\Delta = \{\langle [i], [j] \rangle \mid @_i M j \in \Delta\}$$

Proposition 4.15. *R_K^Δ is well-defined.*

Proof. We have to prove two different properties:

1. On the one hand, if $i \sim_\Delta i'$ and $\langle [i], [j] \rangle \in R_K^\Delta$, then also $\langle [i'], [j] \rangle \in R_K^\Delta$. This is a consequence of Δ being maximally consistent set and the definition of R_K^Δ.

2. On the other hand, if $j \sim_\Delta j'$ and $\langle [i], [j] \rangle \in R_K^\Delta$, then also $\langle [i], [j'] \rangle \in R_K^\Delta$. This is a consequence of Δ being a maximally consistent set, the definition of R_K^Δ and theorem 4.3.

□

Definition 4.16. *Let Δ be a maximal consistent, named and fully-saturated set, let $[i], [j]$ be the equivalence classes of $i, j \in NOM$ (respectively), and let $\delta \in FORM^*$. We define an imagination relation as follows:*

$$R_{Img}^\Delta = \{\langle [i], [j], \delta \rangle \mid @_i\langle I(\delta)\rangle j \in \Delta\}$$

for $\delta \in FORM^*$, δ not being contradictory (that is: $\delta \not\equiv \bot$), and for $i, j \in NOM$.

Proposition 4.17. *R_{Img}^Δ is well defined.*

Proof. Similarly, we must prove that R_{Img}^Δ is well-defined. In this case, it is consequence of Δ being a maximally consistent set, the definition of R_{Img}^Δ, theorem 4.3 and axiom 4b.

□

Definition 4.18. *Let Δ be a maximal consistent, named and fully-saturated set. We define the canonical model of Δ, called $\mathcal{M}_\Delta = \langle W_\Delta, R_K^\Delta, R_{Img}^\Delta, V_\Delta, N_\Delta \rangle$ as follows (where $[i],[j]$ are the equivalence classes of nominals $\{i,j\} \subseteq NOM$, where $\delta \in FORM^*$ and $p \in ATOM$):*

- $W_\Delta = \{[i] \mid i \textit{ is a nominal}\}$
- $R_K^\Delta = \{[i],[j] \mid @_i Mj \in \Delta\}$
- $R_{Img}^\Delta = \{\langle [i],[j],\delta \rangle \mid @_i \langle I(\delta) \in \Delta\}$
- $V_\Delta(p) = \{[i] \in W_\Delta \mid @_i p \in \Delta\}$
- $N_\Delta(i) = \{[i]\}$

$\mathcal{M}_\Delta$ is well-defined and then we can enunciate and prove the Truh Lemma.

Lemma 4.19. *[**Truth Lemma**] For any maximal consistent, named and fully-saturated set Δ, the following statement holds (where $[i]$ is the equivalence class of $i \in NOM$ and φ is a well-formed formula of the language):*

$$\mathcal{M}_\Delta, [i] \vDash \varphi \textit{ if and only if } @_i\varphi \in \Delta$$

Proof. We will prove this lemma by induction on the structure of φ. Since the proofs for the inductive cases $\varphi := j$, $\varphi := p \in$ ATOM, $\varphi := \neg\psi$ and $\varphi := \psi_1 \wedge \psi_2$ are trivial, we will skip them and only sketch the remaining cases.

1. $\varphi := M\psi$: We want to prove that $\mathcal{M}_\Delta, [i] \vDash M\psi$ iff $@_i M\psi \in \Delta$.
   - Suppose $\mathcal{M}_\Delta, [i] \vDash M\psi$. Then there is a world $[j]$ such that $\langle [i],[j] \rangle \in R_K^\Delta$ and $\mathcal{M}_\Delta, [j] \vDash \psi$. Then, $@_j\psi \in \Delta$ by the induction hypothesis. By the definition of R_K^Δ, we also have that $@_i Mj \in \Delta$. Then, by theorem 4.3 it follows that $\Delta \vdash @_i M\psi$ which, being Δ a maximal consistent set, implies that $@_i M\psi \in \Delta$.
   - Suppose now that $@_i M\psi \in \Delta$. Then, by M-saturation, we have that $@_i Mj \in \Delta$ and $@_j\psi \in \Delta$, for some world-nominal j. By definition of R_K^Δ this means that $\langle [i],[j] \rangle \in R_K^\Delta$ and, by the induction hypothesis, we know that $\mathcal{M}_\Delta, [j] \vDash \psi$. Therefore, it follows that $\mathcal{M}_\Delta, [i] \vDash M\psi$.
2. $\varphi := \mathrm{Img}(\delta)$: We want to prove that $\mathcal{M}_\Delta, [i] \vDash \mathrm{Img}(\delta)$ iff $@_i\mathrm{Img}(\delta) \in \Delta$.
   - Suppose that $\mathcal{M}_\Delta, [i] \vDash \mathrm{Img}(\delta)$. Then, $\texttt{ImgAlg}(\delta, [i])$ has been executed and, being $\delta \in FORM^*$ with $\mathrm{DNF}(\delta) = \delta_1 \vee \ldots \vee \delta_m$, there are $[k_1], \ldots, [k_m]$ worlds such that $([i],[k_l],\delta) \in R_{Img}^\Delta$, for $l = 1 \ldots m$. And then, by the definition of the canonical model: $\Delta \vdash (\bigwedge_{l=1\ldots m} @_i\langle I(\delta) \rangle k_l)$. By Axiom 4c we obtain that $\Delta \vdash \mathrm{Img}(\delta)$, and $@_i\mathrm{Img}(\delta) \in \Delta$.
   - Suppose $@_i\mathrm{Img}(\delta) \in \Delta$ ($\mathrm{DNF}(\delta) = \delta_1 \vee \ldots \vee \delta_n$). Since Δ is *Img*-saturated, there are new nominals $k_1, \ldots, k_n \in$ NOM such that for all $l = 1, \ldots, n$ and $r = 1, \ldots, n$: $@_i\langle I(\delta) \rangle k_l \in \Delta$, $@_{k_l} p \in \Delta$ for $p \in PL(\delta_l)$, and $@_{k_l} p \in \Delta$ if $@_i p \in \Delta$ for $p \notin NL(\delta_l)$. By using the definition of the canonical model we get that $\mathcal{M}_\Delta, [i] \vDash \mathrm{Img}(\delta)$.
3. $\varphi := \langle I(\delta) \rangle \varphi$: We want to prove that $\mathcal{M}_\Delta, [i] \vDash \langle I(\delta) \rangle \varphi$ iff $@_i\langle I(\delta) \rangle\varphi \in \Delta$.
   - L Suppose $\mathcal{M}_\Delta, [i] \vDash \langle I(\delta) \rangle \varphi$. By definition, there is some $[j]$ such that $\langle [i],[j],\delta \rangle \in R_{Img}^\Delta$, $\mathcal{M}_\Delta, [i] \vDash \langle I(\delta) \rangle j$ and $\mathcal{M}_\Delta, [j] \vDash \varphi$. Since Δ is maximal consistent, $\Delta \vdash @_i\langle I(\delta) \rangle j \wedge @_j\varphi$ and, by axiom 4b, we get that $\Delta \vdash @_i\langle I(\delta) \rangle \varphi$, and so $@_i\langle I(\delta) \rangle\varphi \in \Delta$.
   - Suppose now that $@_i\langle I(\delta) \rangle\varphi \in \Delta$. By the definition of I-saturated, $@_i\mathrm{Img}(\delta) \in \Delta$ and there is some $j \in$ NOM such that $@_i\langle I(\delta) \rangle j \in \Delta$ and $@_j\varphi \in \Delta$. Then $\langle [i],[j],\delta \rangle \in R_{Img}^\Delta$. By the induction hypothesis on φ, $\mathcal{M}_\Delta, [j] \vDash \varphi$, and then we get that $\mathcal{M}_\Delta, [i] \vDash \langle I(\delta) \rangle\varphi$.

□

5 Discussion

Although our approach is on the right track, it still has certain shortcomings that we would want to amend in a future version. One of the main ones is related to the way the agent imports information not directly specified by the initial premise. As seen in the theories reviewed in Section 2, an important feature of imagination is that most of the information developed in an imaginary scenario is based on reality-oriented rules and facts which, in turn, are believed or known to be true by the agent.

The first shortcoming, with respect to this, is that our formal system misses a way to account for the notion of "reality-oriented rules". We may argue that, once the initial premise is clamped into the new imaginary worlds, the `ImgAlg` looks into the world of reference and imports "reality-oriented facts", being the atomic formulas that describe the state of affairs represented by the world of reference. Nevertheless, importing specific *facts* into the imaginary scenario is not the same as using *rules* to infer what else would be the case in there. Even though we provide an alternative interpretation of the dynamic mechanisms for imagination acts in [9], that alternative approach neither takes into account the epistemic side of the picture, nor it is proved to be sound and complete.

The second shortcoming of our formal system is directly inherited from the fact that the single-agent epistemic logic we took to build our logic upon cannot, at least for now, represent beliefs in an explicit way. Even though we can interpret operator M as a weak form of belief, this interpretation lacks the kind of "preference ordering" which is often attributed to our beliefs, making some of them more plausible than others. Nevertheless, this is the stance we have taken in our proposal regarding the way we use the single-agent epistemic logic, and in order to bypass the lack of an explicit belief operator. By performing an act of imagination at a possible world w, the agent considers, in the resulting imaginary worlds, that her beliefs are those facts represented in world w. This, however, usually forces our agent to imagine more than she is supposed to, as she could end up importing atomic propositions that may not be known by her, but which are seen as known in the imaginary worlds; needless to say, this may do the trick as a first approach, but it is not as accurate as we would like it to be.

Therefore, the solution needed to overcome the issue regarding the mirroring effect would need two major changes in our logic:

1. Expand the system with a new operator B able to explicitly account for "believe" as a modal attitude with some preferred or believed worlds by the agent.

2. Update the third step of the `ImgAlg`: instead of going over the atomic propositions that are true at the world of reference, the algorithm would need to check, for each atomic proposition in the model, which ones are actually *believed* by the agent, and import only those.

There are many alternative ways to represent an agent's beliefs (see [17]), but there are two approaches that are specially interesting for us: either to add a new binary relation to our current setting (as suggested in Chapter 7 of [17]), or to use the so-called *plausibility models* (also in Chapter 7 of [17], or in [3]). Things are seldom as simple as they sound, however, and adding an explicit representation for beliefs in our system is not an exception. Let's briefly consider how our logic would behave if we added plausibility models to it.

Consider a brief example, as depicted in Figure 7. At the left side of the figure we have a plausibility model representing an agent who believes that w is the actual state of affairs, but who nevertheless also considers a different state of affairs v to be possible. As it can be seen in the figure, executing an act of imagination with initial content q at either w or v results, in the end, in the same imaginary world, as the agent always imports her beliefs from world w.

Therefore, we can see how, when using plausibility models or, in fact, any explicit representation for beliefs in our logical system, the world of reference would no longer matter, as the atomic formulas believed by the agent would always be those atoms holding at the same possible world, no matter what[12].

[12] It can also be the case, in plausibility models, that we have not just a single top-world, but rather a set of various top-worlds, which the agent believes to be the case, but over which she has no preference. We will not unfold the technical details of how this case should work, but the `ImgAlg` would probably need to duplicate the same structure of top-worlds in the imagining to represent the fact that the agent's plausibility order over certain facts is not determined.

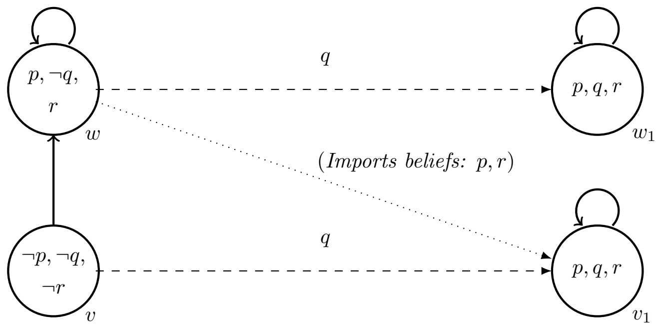

Figure 7: The imported atoms come from the same world as before.

After considering this simple example, we ask ourselves: what would be the role of the world of reference, then, if our Logic of Imaginary Scenarios could account for an explicit representation of beliefs? Moreover, and if the agent's beliefs were always taken from the same real possible world, would the belief structure represented by the real possible worlds even matter, in our system? If the imported atomic formulas were always taken from the same real possible world, why would we even consider any real possible world different from the top-world?

Taking this into account, we argue that, although adding an explicit representation for beliefs is worth considering, our current approach to the Logic of Imaginary Scenarios, in which we have chosen to account for a weak form of beliefs, is, after all, an approach that gives us more flexibility, in terms of exploring the different acts of imagination the agent could perform. This is not to say that we could not include explicit beliefs as part of our system, but we have just shown that, in such cases, we would also need to come up with a way of avoiding a kind of "non-top-world-trivialization" or, in other words, of avoiding that the presence of possible worlds that are not among the agent's explicit beliefs becomes trivial.

A concern that stems from this, however, is that there are cases in which the agent may end up importing more than she should —particularly, atomic formulas that are not known by the agent would still be imported by the way the `ImgAlg` works. Nevertheless, this concern could be alleviated if we reinterpret our reading of imagination acts as "the agent imagines δ while considering that the actual state of affairs is represented by the world of reference w". By allowing this reading, we implicitly require to our agent to commit to one of her believed worlds before imagining, but, at the same time, it provides enough flexibility as for the agent to explore how different believed worlds would accommodate a similar imaginary scenario.

6 Conclusions and Future Work

The main goal of the present work was to define a dynamic formal system to capture, through the execution of an algorithm that expands their formal models, how an agent creates new imaginary scenarios as a result of executing a voluntary act of imagination. The Logic of Imaginary Scenarios does that and models a single-agent setting able to handle the basics of epistemic logic, while adding the required structure to account for imagination acts. In particular, our system captures the so-called voluntary mode of the kind of imagination acts studied in the former theories by clamping the initial content δ of the imagining as one of the arguments of the `ImgAlg`; furthermore, the involuntary mode is then captured by the way the algorithm imports atomic formulas left unspecified by δ, while giving priority to the initial content. The way the algorithm expands the formal models allows this logic to be truly dynamic, in the sense that it allows the agent represented in it to imagine any δ that fits the required form and expand the formal model accordingly. In this sense, the Logic of Imaginary Scenarios is on the right track, and provides a valuable step forward with respect to other existing formal systems which fail to capture the dynamic aspect of this mental attitude. Furthermore, our logic is sound and complete with respect to the relevant class of models, which provides a solid dynamic formal system for imaginary scenarios to build upon and further expand with even more nuanced features.

As we already hinted in Section 5, a natural line of future work is to expand our system in order to account for a more detailed representation for beliefs; as we have seen, though, this is not devoid of challenges that will affect both the models and the Imagination Algorithm. Furthermore, it would be interesting to lift up the restriction on imagining only propositional formulas, thus allowing the agent to imagine something about her beliefs, for instance, or to allow for nested imagination acts. Both lines of future work would likely require quantification over nominals in order to deal not only with the content of specific worlds, but also with their relational structure. Additionally, considering paraconsistency and paracompleteness would allow to consider whether to import every known and believed facts into a new imaginary scenario, or only those that are somehow relevant with the initial content defining the imagining; still in this line, considering Berto's work on aboutness in [5] could prove particularly useful.

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